A Lax Pair for the 2D Euler Equation

Yanguang (Charles) Li *

Department of Mathematics,

University of Missouri

Columbia, MO 65211

E-mail: cli@math.missouri.edu

February 1, 2008

*This work is supported by the Guggenheim Fellowship.

Abstract

A Lax pair for the 2D Euler equation is found.

PACS Codes: 47, 02.
MSC numbers: 35, 51.
Keywords: Lax Pair, Euler Equation.

1. A Lax Pair for the 2D Euler Equation

This is to report that a Lax pair for the 2D Euler equation is found. We write the 2D Euler equation in the vorticity form,

$$\frac{\partial \Omega}{\partial t} + \{\Psi, \Omega\} = 0 \ , \tag{1.1}$$

where Ω is the vorticity, Ψ is the stream function, and the bracket $\{ \ \}$ is defined as

$$\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g) \ .$$

Let us denote the x-directional and the y-directional velocities by u and v respectively. Then

$$u = -\frac{\partial \Psi}{\partial y} \ , \quad v = \frac{\partial \Psi}{\partial x} \ , \quad \Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \ , \quad \Delta \Psi = \Omega \ .$$

The Lax pair is given as

$$\begin{cases} L\varphi = \lambda \varphi \ , \\ \partial_t \varphi + A\varphi = 0 \ , \end{cases} \tag{1.2}$$

where

$$L\varphi = \{\Omega, \varphi\} \ , \quad A\varphi = \{\Psi, \varphi\} \ ,$$

and λ is a complex constant, and φ is a complex-valued function. The compatibility condition of the Lax pair (1.2) gives the 2D Euler equation (1.1), i.e.

$$\partial_t L = [L, A] \ ,$$

where $[L, A] = LA - AL$, gives the Lax representation of the 2D Euler equation (1.1).

Remark 1.1 *With the recent development on chaos in partial differential equations [1] [2] [3], I am interested in building a dynamical system theory for 2D Euler equation under periodic boundary condition [4] [5]. In particular, I am investigating the existence v.s. nonexistence of homoclinic structure. For such studies, it will be fundamentally important to find a Lax pair (if it exists) for the 2D Euler equation. Then I started with Vladimir Zakharov's paper [6]. Zakharov proposed the Lax pair*

$$\begin{cases} \lambda D_1 \varphi + \{\Omega, \varphi\} = 0 \ , \\ \partial_t \varphi + \lambda D_2 \varphi + \{S, \varphi\} = 0 \ , \end{cases}$$

where

$$D_1 = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \ , \quad D_2 = \gamma \frac{\partial}{\partial x} + \delta \frac{\partial}{\partial y} \ ,$$

$\alpha, \beta, \gamma,$ and δ are real constants, λ is a complex constant, S is a real-valued function, and φ is a complex-valued function. The compatibility condition of this Lax pair gives the following equation instead of the 2D Euler equation,

$$\begin{cases} \frac{\partial \Omega}{\partial t} + \{S, \Omega\} = 0 \ , \\ \\ D_1 S = D_2 \Omega \ . \end{cases}$$

(Notice the misprints in the English translation of the article [6].)

Remark 1.2 *The author is also aware of the Lax pair in the inverse Cauchy-Green tensor variable of the Lagrangian formulations of both 2D and 3D Euler equations found by Susan Friedlander and Misha Vishik [7] [8].*

References

[1] Y. Li and D. W. McLaughlin. Morse and Melnikov Functions for NLS Pde's. *Commun. Math. Phys.*, 162:175–214, 1994.

[2] Y. Li, D. McLaughlin, J. Shatah, and S. Wiggins. Persistent Homoclinic Orbits for a Perturbed Nonlinear Schrödinger equation. *Comm. Pure Appl. Math.*, XLIX:1175–1255, 1996.

[3] Y. Li. Smale Horseshoes and Symbolic Dynamics in Perturbed Nonlinear Schrödinger equations. *J. Nonlinear Sci.*, 9:363–415, 1999.

[4] Y. Li. On 2D Euler Equations. I. On the Energy-Casimir Stabilities and the Spectra for Linearized 2D Euler Equations. *J. Math. Phys.*, 41, no. 2:728–758, 2000.

[5] Y. Li. On 2D Euler Equations. II. Degeneracy v.s. Nondegeneracy of the Hyperbolic Foliations - A Galerkin Truncation Study. *submitted, Physica D*, 2000.

[6] V. E. Zakharov. On the Algebra of Integrals of Motion in Two-Dimensional Hydrodynamics in Clebsch Variables. *Functional Anal. Appl.*, 23, no. 3:189–196, 1989.

[7] S. Friedlander and M. Vishik. Lax Pair Formulation for the Euler Equation. *Phys. Lett. A*, 148, no. 6-7:313–319, 1990.

[8] M. Vishik and S. Friedlander. An Inverse Scattering Treatment for the Flow of an Ideal Fluid in Two Dimensions. *Nonlinearity*, 6, no. 2:231–249, 1993.